

Contributions
to the Hierarchy of
Bimodal Singularities

Diploma Thesis

by

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Bonn 1984

Contents

Introduction	5
I Results from the Theory of Isolated Hypersurface Singularities	12
I.1 Basic Concepts and Properties	13
I.1.1 Jet Spaces, Right Equivalence, and Contact Equivalence	13
I.1.2 Isolated Singularities	13
I.1.3 Corank, Multiplicity, and Tangent Cone	13
I.1.4 Quasihomogeneous and r -Nondegenerate Function Germs	13
I.2 Unfoldings, Deformations, Transversal Slices, μ -Homotopy, μ -Adjacency, and Modality	13
I.2.1 Unfoldings of Function Germs	13
I.2.2 Deformations of Complex Space Germs	13
I.2.3 Transversal Slices	13
I.2.4 μ -Homotopy and μ -Adjacency	13
I.2.5 Modality	13
I.3 Arnold's Classification of Singularities of Modality at Most Two	13
I.4 Topological Invariants and Their Behaviour under μ -Homotopy	13
I.4.1 Dynkin Diagrams	13
I.4.2 Spectra and a Conjecture of Arnold	13
I.4.3 Resolution of Singularities and Their Invariants under μ -Homotopy	13
I.4.3.1 Plane Curve Singularities	13
I.4.3.2 Two-Dimensional Hypersurface Singularities	13
I.5 Arnold Classes and μ -Homotopy	13
I.5.1 Supplement to a Proof	13
I.5.2 Modality and μ -Homotopy	13
II The Problem of This Work and the Structure of the Proofs of the Main Results	14
II.1 The Notion of Adjacency, Known Adjacency Results, and the Problem of This Work	14
II.1.1 The Notion of Adjacency	14
II.1.2 Results on Adjacencies between Unimodal and Bimodal Singularities due to Karras, Brieskorn, and Laufer	14
II.1.3 The Problem of This Work	14
II.2 The Structure of the Argument	14
II.2.1 A Rough Structural Description of Adjacency Behaviour	14
II.2.2 Arguments Using the Tjurina Number	14

II.2.3	The Investigation of Topological Invariants and Spectra	14
II.2.4	On the Analysis of the Equations	14
III	Results	15
IV	Methods	16
IV.1	Primitive Embeddings of Milnor Lattices	17
IV.1.1	Integral Lattices and Discriminant Forms	17
IV.1.2	The Invariants (s_+, s_-, s'_0, q) of a Milnor Lattice	17
IV.1.3	Existence and Uniqueness of Even Nondegenerate Lattices with Invariants (s_+, s_-, q)	17
IV.1.4	Overlattices and Primitive Embeddings of Even Nondegenerate Lattices	17
IV.1.5	Primitive Embeddings of Degenerate Even Lattices	17
IV.1.6	On the Application	17
IV.2	The Investigation of the Transversal Slice	17
IV.2.1	Construction of Versal Unfoldings and Transversal Slices	17
IV.2.2	\mathbb{C}^* -Action	17
IV.2.3	A Result of Wirthmüller	17
IV.2.4	Independence of the Stratification of T from the Chosen Minimal Transversal Slice	17
IV.2.5	Concluding Remarks	17
IV.3	Plane Curve Singularities and Reduction from Surface Singularities to Curve Singularities	17
IV.3.1	Isolated Plane Curve Singularities	17
IV.3.1.1	The Standard Resolution	17
IV.3.1.2	The System of Multiplicity Sequences	17
IV.3.1.3	The Multiplicity Tree	17
IV.3.1.4	A Procedure for Deriving Algebraic Conditions	17
IV.3.2	Reduction of the Investigation of Surface Singularities to Curve Singularities	17
V	Computations	18
V.1	Primitive Embeddings into Milnor Lattices of the Groupings E, Z, Q, W, S, U	18
V.1.1	Grouping E	18
V.1.2	Grouping Z	18
V.1.3	Grouping Q	18
V.1.4	Grouping W	18
V.1.5	Grouping S	18
V.1.6	Grouping U	18
V.2	Analysis of the Equations	18
V.2.1	Investigation of the Classes $J_{3,0}, E_{18}, E_{19}, E_{20}$	18
V.2.2	Investigation of the Classes $Z_{1,0}, Z_{17}, Z_{18}, Z_{19}$	18
V.2.2.1	General Part	18
V.2.2.2	Investigation of $Z \rightarrow E$	18
V.2.2.2.1	Derivation of the Equations	18

V.2.2.2.2	Analysis of the Equations	18
V.2.3	Investigation of the Classes $Q_{2,0}, Q_{16}, Q_{17}, Q_{18}$	18
V.2.3.1	General Part	18
V.2.3.2	Investigation of $Q \rightarrow E$	18
V.2.3.2.1	Derivation of the Equations	19
V.2.3.2.2	Analysis of the Equations	19
V.2.4	Investigation of the Classes $W_{1,0}, W_{17}, W_{18}$	19
V.2.4.1	General Part	19
V.2.4.2	Investigation of $W \rightarrow E$	19
V.2.4.2.1	Derivation of the Equations	19
V.2.4.2.2	Analysis of the Equations	19
V.2.5	Investigation of the Classes $S_{1,0}, S_{16}, S_{17}$	19
V.2.5.1	General Part	19
V.2.5.2	Investigation of $S \rightarrow E$	19
V.2.5.2.1	Derivation of the Equations	19
V.2.5.2.2	Analysis of the Equations	19
V.2.5.3	Investigation of $S \rightarrow Z, W$	19
V.2.5.3.1	Derivation of the Equations	19
V.2.5.3.2	Analysis of the Equations	19
V.2.6	Investigation of the Classes $U_{1,0}$ and U_{16}	19
V.2.6.1	Preparation for the Analysis of the Equations	19
V.2.6.2	Investigation of $U_{1,0} \rightarrow T_{2,38}/T_{2,39}$	19
V.2.6.2.1	Derivation of the Equations	19
V.2.6.2.2	Analysis of the Equations	19
V.2.6.3	Investigation of $U \rightarrow Z, W$	19
V.2.6.3.1	Derivation of the Equations	20
V.2.6.3.2	Analysis of the Equations	20
V.2.6.4	Investigation of $\text{not } (U_{16} \xrightarrow{\exists} T_{3,38})$	20
V.2.6.5	Additional Group Action on Surface Singularities with \mathbb{C}^* -Action	20

APPENDIX A **21**

A.1	Normal Forms	21
A.2	Table of the Adjacency Results for Theorems 1 and 2 of Chapter III	21
A.3	Adjacency Results of Laufer	21
A.4	The Relations $L(Y) \rightarrow L(X), \mathcal{D}_s(Y) \leq_s \mathcal{D}_s(X), \mathcal{D}(Y) \leq \mathcal{D}(X), \text{Spec}(X) \rightarrow \text{Spec}(Y)$	21
A.5	List of Continuous Families	21
A.6	Signatures and Discriminant Forms	21
A.7	Systems of Multiplicity Sequences	21
A.8	Spectra	21

APPENDIX B **22**

B.1	Computer Program	22
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Bibliography **23**

Introduction

A hierarchy within the class of isolated singularities of holomorphic function germs is defined by the simplification relation. We say that an isolated singularity $f \in \mathfrak{m}^2(n)$ simplifies (or deforms) to an isolated singularity $g \in \mathfrak{m}^2(n)$ (written $f \xrightarrow[\mu]{} g$) if, by a small perturbation of f , one can obtain singularities which are μ -homotopic to g . (The notion of μ -homotopy is a rather coarse equivalence relation compared to right- and contact-equivalence.)

No general method is known which allows one to decide whether a simplification exists between two arbitrarily given singularities. All previous work on simplifications between isolated hypersurface singularities is restricted to certain classes lying at the lowest levels of the hierarchy.

A measure for the position of a singularity within this hierarchy is given by its modality. Arnold's classification (cf. [Ar3]) comprises all isolated hypersurface singularities of modality at most two. The classes of modality 0, 1, and 2 are each subdivided into infinite series and exceptional classes. In both the unimodal and bimodal cases, there arise fourteen exceptional classes which exhibit many similarities.

E. Brieskorn investigated the simplifications of the singularities belonging to the fourteen exceptional unimodal classes within the unimodal category ([Br2]). He showed how this behaviour is reflected in purely arithmetic properties of their Milnor lattices ([Br3]).

Some simplifications of singularities from the exceptional bimodal classes are known from the work of H. Laufer ([Lau3]). In the present work we continue the investigation of the simplification behaviour of these classes, following the methods developed by Brieskorn in [Br2].

One of the main results of this work is the determination of all simplifications of the singularities from the exceptional bimodal classes

$$\begin{array}{lll} E_{18}, E_{19}, E_{20} & W_{17}, W_{18} & U_{16} \\ Z_{17}, Z_{18}, Z_{19} & S_{16}, S_{17} & \\ Q_{16}, Q_{17}, Q_{18} & & \end{array}$$

into unimodal and bimodal singularities.

We show that— analogously to the simplification behaviour of unimodal singularities—these simplifications do not depend on the choice of representatives of the exceptional classes. That is, if a singularity from a class X simplifies to singularities of a class Y (written $X \xrightarrow{\exists} Y$), then this holds for all singularities in X (written $X \xrightarrow{\forall} Y$).

The simplifications obtained are collected in a table which, due to its size, is reproduced both following this introduction and in tabular form in Appendix A.

It is not to be expected that the different simplification behaviour of the exceptional bimodal classes can be described, as in the unimodal case, in terms of arithmetic properties

of the Milnor lattices. Indeed, as shown by W. Ebeling, the singularities E_{18} and Q_{18} , for example, have isomorphic Milnor lattices.

There exists, however, a stronger invariant, namely the set of distinguished Dynkin diagrams associated with an isolated hypersurface singularity, which allows one to distinguish between the classes of modality 0, 1, and 2. These diagrams describe the intersection behaviour of a geometrically distinguished basis of the Milnor lattice.

It is known that if a singularity f simplifies to a singularity g , then every distinguished Dynkin diagram of g can be embedded as a subgraph into a distinguished Dynkin diagram of f . It is therefore natural to ask under which conditions such embeddings occur and whether they correspond to simplifications. However, the set of distinguished Dynkin diagrams of a singularity appears to be highly complicated, and this question is therefore difficult. Based on his investigations in the unimodal case, Brieskorn conjectured that such a subgraph relation always implies a simplification ([Br4]). He was able to verify this conjecture in several cases.

If one compares the embedding behaviour of certain distinguished Dynkin diagrams considered by Gabrielov and Ebeling with the simplifications studied here, one finds that although there exist simplifications not reflected by diagram embeddings, conversely every such embedding gives rise to a simplification.

In the course of studying embeddings for further bimodal classes, we encountered examples of the following type:

There exists a distinguished Dynkin diagram of the unimodal class E_{14} which can be extended to one of the bimodal class $W_{1,0}$. On the other hand, a comparison of the Tjurina numbers shows that $W_{1,0} \not\rightarrow E_{14}$.

These observations led us to investigate the simplification behaviour of the lowest bimodal series classes in more detail. In contrast to the exceptional classes, this investigation turns out to be simpler, with the exception of the class $U_{1,0}$.

A second main result of this work is the determination of all simplifications of singularities from the bimodal series classes

$$J_{3,0}, Z_{1,0}, Q_{2,0}, W_{1,0}, S_{1,0}, U_{1,0}$$

into unimodal and bimodal singularities.

In this case there occur precisely four simplifications for which

$$X \xrightarrow{\exists} Y, \quad \text{but not } X \xrightarrow{\forall} Y.$$

An important further invariant of an isolated hypersurface singularity is the spectrum, a tuple of rational numbers closely related to the eigenvalues of the monodromy and defined via a mixed Hodge structure.

According to a conjecture of Arnold ([Ar4]), a simplification implies a certain adjacency relation between the corresponding spectra. In all cases considered in this work, this conjecture is confirmed; it has recently been proved by J.H.M. Steenbrink.

Methods

a) Invariants

Most results of the form $\text{not}(X \rightarrow Y)$ are obtained by studying invariants. Simple invariants such as corank and Milnor number already restrict the possible simplifications considerably. A necessary condition for a simplification is the existence of a primitive embedding of Milnor lattices. This can be decided using their discriminant forms, following Nikulin [Ni].

For the computation of the discriminant forms we use the work [Eb2] of W. Ebeling. Ebeling provides Dynkin diagrams as well as signatures and decompositions of the Milnor lattices for most of the singularities classified by Arnold. From these invariants the discriminant forms can be computed easily.

b) Analysis of equations

The cases in which $X \rightarrow Y$ cannot be excluded by considering invariants, as well as results of the form $X \xrightarrow{\exists} Y$, $X \xrightarrow{\forall} Y$, and results of the form $X \xrightarrow{\exists} Y$ but not $X \xrightarrow{\forall} Y$, are proved by establishing, for suitable $f \in X$, $g \in Y$, deformations $f \rightarrow g$ via an analysis of the equations.

For this purpose it suffices to study a transversal slice

$$T: \mathbb{C}^r \longrightarrow \mathfrak{m}^2(n)$$

with

$$T(t) = f + \sum_{i=1}^r t_i \psi_i, \quad \psi_1, \dots, \psi_r \text{ generate } \mathfrak{m}^2/\mathfrak{m}\Delta f,$$

to the right orbit of f .

The problem of determining for which parameters t the singularities $T(t)$ are μ -homotopic to g leads to algebraic conditions on the parameters $t \in \mathbb{C}^r$. If g is stably equivalent to a curve singularity, we derive these conditions using the system of multiplicity sequences of this curve singularity. In the other cases considered here, g is stably equivalent to a surface singularity of multiplicity three. In these cases we may reduce the problem to the study of curve singularities by a method due to C.T.C. Wall.

The computation of explicit solutions of the algebraic conditions yields continuous families f_t with $f_0 = f$ and $f_t \sim_{\mu} g$ for $t \neq 0$. Many adjacencies $f \rightarrow g$ already follow from the existence of such continuous families. For quasihomogeneous f , the analysis can be considerably simplified by using transversal slices with a \mathbb{C}^* -action.

Structure of the work:

In Chapter I we collect the basic concepts and results from the theory of isolated hypersurface singularities that are needed for our investigation. These are largely known, although they are scattered throughout the literature and not presented uniformly in the various sources. Chapter I is therefore rather detailed.

Chapters II and III are central. Here the reader finds the precise formulation of the problem, the main results of the work, as well as known adjacency results insofar as they are related to our investigation. Furthermore, the structure of the proofs of the main results is explained and the more general, shorter parts of the proofs are carried out.

The methods for the investigation of primitive embeddings of Milnor lattices, the study of transversal slices, and the derivation of algebraic conditions are presented in Chapter IV.

Chapter V is divided into two independent parts. In V.1 we study primitive embeddings of Milnor lattices in order to exclude adjacencies between singularities. In V.2 we prove,

by an analysis of the equations, statements of the form $f \not\rightarrow g$ that cannot be established by considering invariants, as well as statements $f \rightarrow g$ that do not follow from the continuous families constructed by us.

As can be seen from the table of contents, each of the investigations in V.2 proceeds in three steps. First, transversal slices are computed and the reduction to curve singularities is carried out (general part or preparation for the analysis of the equations).

In the next step, algebraic conditions are derived simultaneously for several classes of singularities (derivation of the equations). Finally, in the last step, we investigate the solution behaviour of the resulting systems of equations (analysis of the equations).

The tables containing the results from Chapter III and the tables of normal forms and invariants of Arnold's classes are collected in Appendix A.

The computer program given in Appendix B was used for deriving the algebraic conditions in Chapter V.2.

We conclude by pointing out some further noteworthy aspects of the work.

- Arnold's classification is discussed in I.3.
- The singularities within an Arnold class are μ -homotopic. In I.2.4 we discuss μ -homotopy and μ -adjacency as well as their properties. In I.5 we show that different Arnold classes lie in different μ -homotopy classes. We also address the question whether the modality is constant along μ -homotopy classes.
- In II.4.1 we give a detailed proof of the invariance of the set of distinguished Dynkin diagrams under μ -homotopy.
- In IV.1.5 we study primitive embeddings of even lattices with one-dimensional kernel into even nondegenerate lattices.
- On monomial transversal slices of quasihomogeneous singularities there is a natural \mathbb{C}^* -action. From this we derive in IV.2.2 several consequences which considerably simplify the analysis of the equations in V.2. We also transfer two results on semiuniversal deformations with \mathbb{C}^* -action to transversal slices with \mathbb{C}^* -action satisfying an additional condition.
- In V.2.6 we briefly discuss an additional group action for the quasihomogeneous singularity from U_{16} . Furthermore, for two singularities from $U_{1,0}$ distinguished by their adjacency behaviour, we give alternative equations possessing an additional symmetry.
- Between certain classes of surface singularities and certain classes of curve singularities a bijection is defined via the reduction method of C.T.C. Wall. Using methods from his work [Wa], we show that this bijection respects the adjacency relation (IV.3.2).

The investigation of adjacencies in this diploma thesis is divided as follows:

Richard studied the adjacencies of the classes

$J_{3,0}, E_{18}, E_{19}, E_{20}$
 $Z_{1,0}, Z_{17}, Z_{18}, Z_{19}$
 $Q_{2,0}, Q_{16}, Q_{17}, Q_{18}$

Dieter studied the adjacencies of the classes

$W_{1,0}, W_{17}, W_{18}$
 $S_{1,0}, S_{16}, S_{17}$

and Franz studied the adjacencies of the classes

$U_{1,0}, U_{16}$

Originally, these were three separate topics. However, the nature of the problem made it natural to combine them into a joint work.

This required intensive discussions and revisions of the drafts due to the different approaches taken. Both in structure and in methods we have aimed at a coherent presentation. This applies in particular to Chapter V.

We would like to thank J. Wahl, K. Wirthmüller, C.G. Gibson, U. Karras and many others for helpful discussions, as well as the computing center RHRZ for providing access to its facilities. Franz would also like to thank the Studienstiftung des deutschen Volkes for financial support.

Our special thanks go to Dr. W. Ebeling for his excellent supervision, to Prof. E. Brieskorn for suggesting this problem and for valuable insights, and to our parents for their support.

Bonn, May 1, 1984

The three B's

	$J_{3,0}$	E_{18}	E_{19}	E_{20}	$Z_{1,0}$	Z_{17}	Z_{18}	Z_{19}	$Q_{2,0}$	Q_{16}	Q_{17}	Q_{18}	$W_{1,0}$	W_{17}	W_{18}	$S_{1,0}$	S_{16}	S_{17}	$U_{1,0}$	U_{16}	
U_{16}																				•	
$U_{1,0}$																				•	
U_{12}																			•	•	
S_{16}																		•			
$S_{1,2}^{\#}$																		•			
$S_{1,1}^{\#}$																	•	•			
$S_{1,2}$																		•			
$S_{1,1}$																	•	•		•	
$S_{1,0}$																	•	•		•	
S_{12}																•	•	•	•	•	
S_{11}																•	•	•	•	•	
W_{17}															•						
$W_{1,2}^{\#}$															•						
$W_{1,1}^{\#}$														•	•						
$W_{1,2}$															•						
$W_{1,1}$														•	•				•		
$W_{1,0}$														•	•				•		
W_{13}													•	•	•			•	•		
W_{12}													•	•	•	•	•	•		•	
Q_{17}												•									
Q_{16}											•	•							•		
$Q_{2,3}$												•									
$Q_{2,2}$											•	•									
$Q_{2,1}$										•	•	•						•	•		
$Q_{2,0}$										•	•	•						•	•	•	
Q_{12}									•	•	•	•				•	•	•	•	•	
Q_{11}									•	•	•	•				•	•	•	•	•	
Q_{10}									•	•	•	•				•	•	•	•	•	
Z_{18}								•													
Z_{17}							•	•							•						
$Z_{1,3}$								•													
$Z_{1,2}$								•	•												
$Z_{1,1}$							•	•	•					•	•				•		
$Z_{1,0}$							•	•	•	•				•	•				•		
Z_{13}					•	•	•	•					•	•	•			•	•	Δ	•
Z_{12}					•	•	•	•					•	•	•	•	•	•	•	•	•
Z_{11}					•	•	•	•					•	•	•	•	•	•	•	•	•
E_{19}				•																	
E_{18}				•	•			•													
$J_{3,3}$				•																	
$J_{3,2}$				•	•																
$J_{3,1}$			•	•	•			•	•			•									
$J_{3,0}$			•	•	•			•	•			•									
E_{14}	•	•	•	•		•	•	•		•	•	•	Δ	•	•				•		•
E_{13}	•	•	•	•	•	•	•	•		•	•	•	•	•	•		Δ	•	•		•
E_{12}	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•

• : $X \overset{\forall}{\rightarrow} Y$
 Δ : $X \overset{\exists}{\rightarrow} Y$, but not $X \overset{\forall}{\rightarrow} Y$

Table 1: The complete list of adjacencies from exceptional bimodal singularity classes to the bimodal singularity classes and the exceptional unimodal classes (see Theorems 1 and 2 of Chapter III).

Results of type $X \overset{\vee}{\rightarrow} T_{P,Q,R}$:

X	$(P, Q, R) \leq$
U_{16}	(5, 5, 5); (4, 4, 7); (2, 4, 9)
$U_{1,0}$	(4, 4, 5); (3, 4, 6); (2, 4, 7); (2, 3, 8)
S_{17}	(3, 6, 6); (3, 5, 8); (3, 4, 9); (2, 5, 9); (2, 3, 10)
S_{16}	(3, 5, 7); (3, 4, 8); (2, 4, 9); (2, 5, 8)
$S_{1,0}$	(3, 5, 5); (3, 4, 6); (3, 3, 7)
W_{18}	(2, 7, 7); (2, 6, 9); (2, 5, 10)
W_{17}	(2, 6, 8); (2, 5, 9); (2, 3, 11)
$W_{1,0}$	(2, 6, 6); (2, 5, 7); (2, 4, 8)
Q_{18}	(3, 3, 11)
Q_{17}	(3, 3, 10); (2, 3, 11)
Q_{16}	(3, 3, 9)
$Q_{2,0}$	(3, 3, 7)
Z_{19}	(2, 4, 12); (2, 3, 13)
Z_{18}	(2, 4, 11); (2, 3, 12)
Z_{17}	(2, 4, 10); (2, 3, 11)
$Z_{1,0}$	(2, 4, 8)
E_{20}	(2, 3, 14)
E_{19}	(2, 3, 13)
E_{18}	(2, 3, 12)
$J_{3,0}$	(2, 3, 10)

An adjacency $X \overset{\vee}{\rightarrow} T_{P,Q,R}$ exists precisely when the triple (P, Q, R) is smaller than or equal to a triple listed for X on the right.

Result: $U_{1,0} \overset{\exists}{\rightarrow} T_{2,3,9}$, but not $U_{1,0} \overset{\vee}{\rightarrow} T_{2,3,9}$.

Table 2: The complete list of all adjacencies of an exceptional bimodal class of singularities into a unimodal class of singularities of type $T_{P,Q,R}$ (see Theorems 1 and 2 of Chapter III).

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Contributions of the Authors

Introduction: Dieter

Chapter I

I.1 Dieter;
Proof of the Lemma in in 1.2.3: Franz

I.2 Franz;
2.5 by Dieter

I.3 Dieter

I.4 as well as the checking of the spectra by ;
4.3: Franz

I.5 Franz

Chapter II: Dieter;

II.2.2 Franz

II.2.4 Dieter and Franz

Chapter III: Formulierung der Ergebnisse von Franz;

Satz 7 von Franz;

Computer calculations of the adjacencies of the spectra by Richard

Chapter IV

IV.1.1 Richard

IV.1.2 Dieter and Richard

IV.1.3 Dieter

IV.1.4–1.6 Franz

IV.2 Franz

IV.3.1 Richard;
Remark (ii) on page 200 by Dieter

IV.3.2 Dieter

Chapter V

In Chapter V all calculations for the groupings

- E, Z, Q are by Richard
- W, S are by Dieter
- U are by Franz

Bibliography: prepared by Franz

Anhang A: All tables made by Richard

Anhang B: Richard